Presentation of problem T3 (9 points): The Greenlandic Ice Sheet

Basic themes

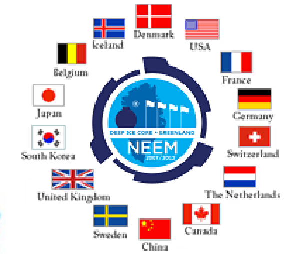
- **Glacier profile**
  - Force-balance
  - Volume vs. Area
- **Ice flow**
  - Horizontal and vertical flow speeds
  - Chronology of ice layers
- **Sea level rise from melting**
  - Melting Greenland vs. Antarctica
  - Gravitational effect?

Drilling ice cores in Greenland

44th IPhO Copenhagen Denmark 7-15 July 2013

Centre for Ice and Climate
Niels Bohr Institute

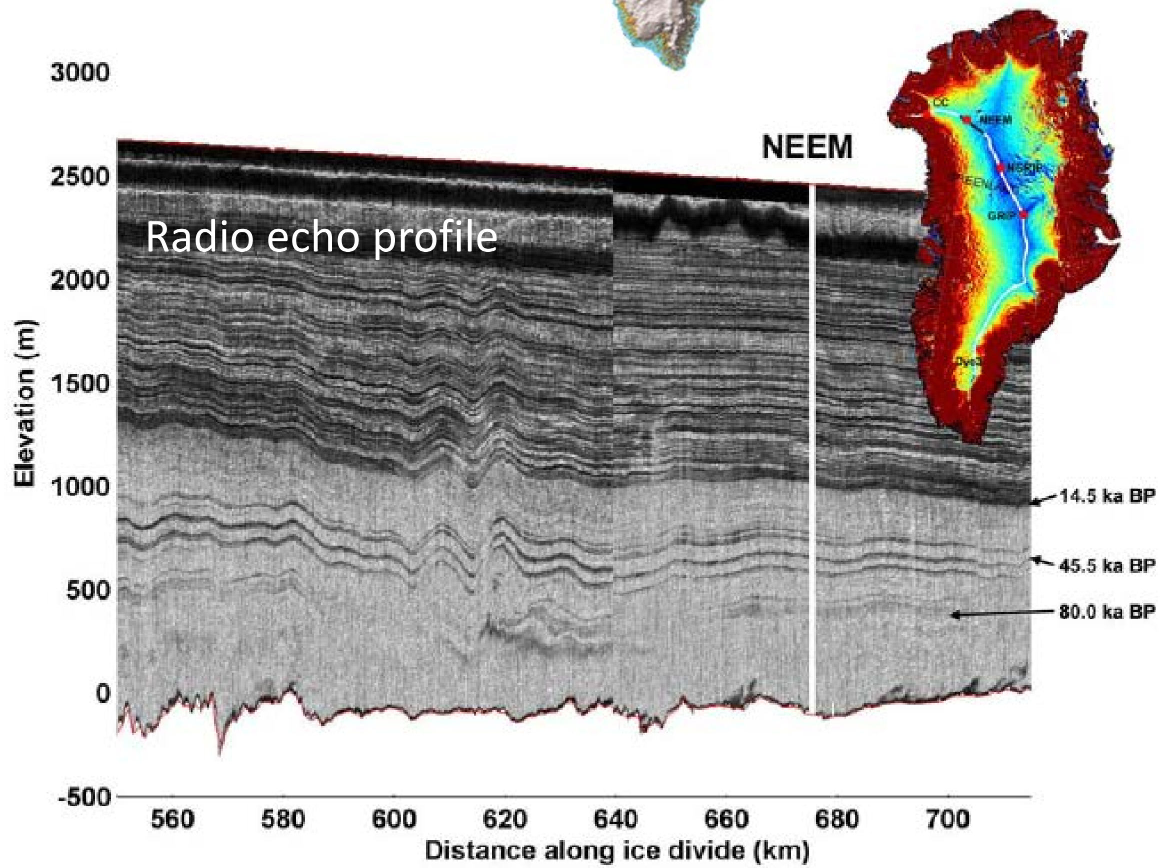
NEEM: an international ice core research project aimed at retrieving an ice core from North-West Greenland reaching back through the previous interglacial, the Eemian.



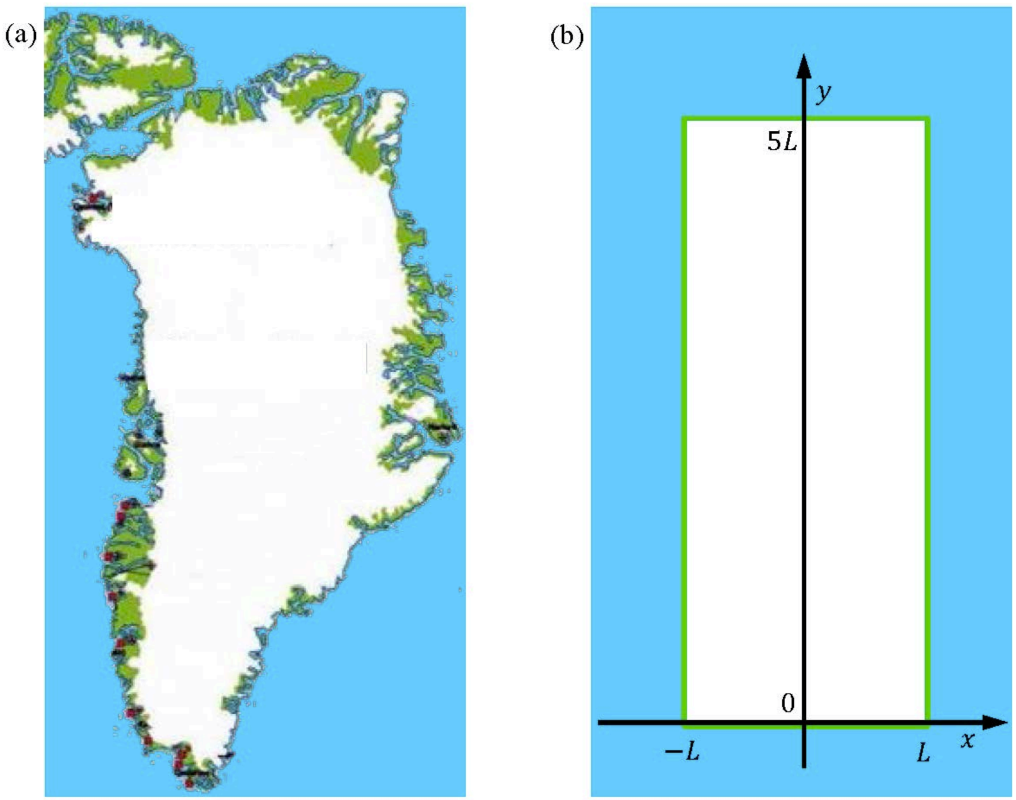
Willi Dansgaard (1922-2011)

37,36 m from bedrock

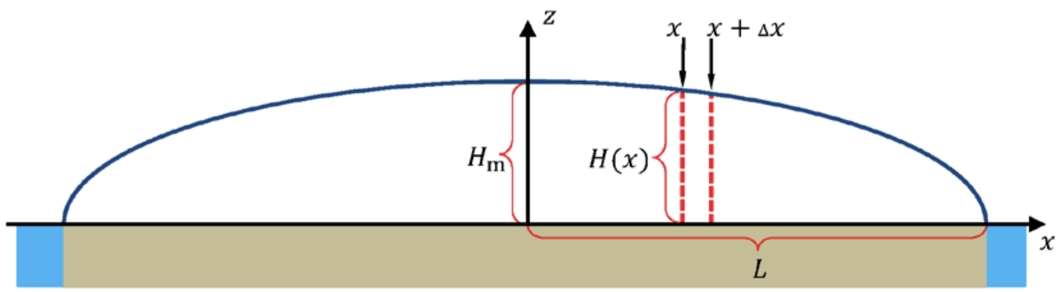
Nature **493**, 489–494 (January 2013)



Simple model of Greenlandic ice-sheet



- **Rectangular ice-sheet**
  - Aspect ratio 2:5
  - Correct Greenland ice area
- **Ice divide along y-axis**
  - Ice flow only in xz-plane
  - No flow parallel to y-axis
- **Height profile**
  - Constant along y-axis
  - Maximum at ice-divide



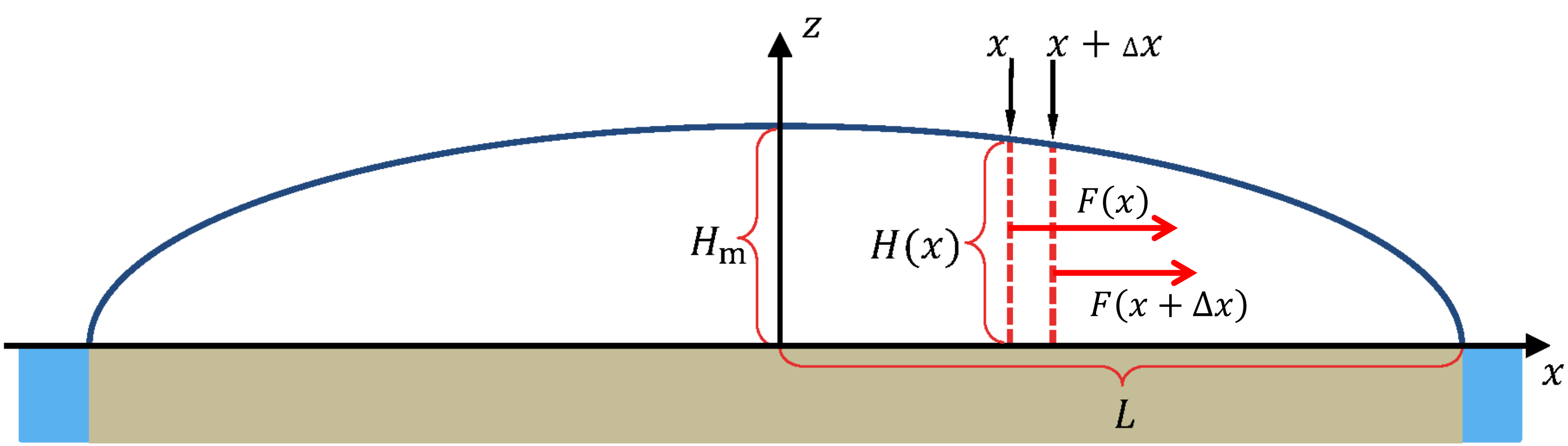
The height profile of the ice sheet

- Treat the ice as an incompressible hydrostatic system
- For fixed height profile $H(x)$ find the (hydrostatic) pressure inside glacier:

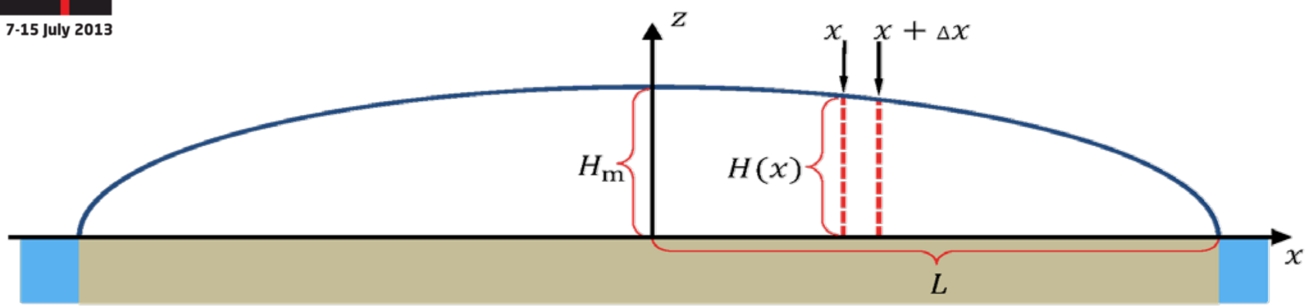
$$p = \rho_{\text{ice}} g (H(x) - z)$$

- Outward force on vertical slice at x of width, $w = \Delta y$, is obtained by integrating up pressure times vertical area:

$$F(x) = \int_0^{H(x)} p(x, z)\, w\, \mathrm{d}z = \frac{1}{2} w\, \rho_{\text{ice}}\, g\, H(x)^2$$



The height profile of the ice sheet



- Net horizontal force component ΔF on the two vertical sides of a slab *(arising from difference in height)*.
- Balanced by friction force $S_\mathrm{b}\,\Delta x \Delta y$ from the ground on the base area $\Delta x \Delta y$, (*basal shear stress* $S_\mathrm{b} = 100\ \mathrm{kPa}$).

$$F(x) = \frac{1}{2}\Delta y\,\rho_\mathrm{ice}\,g\,H(x)^2 \rightarrow S_\mathrm{b}\,\Delta x \Delta y = \Delta F(x) = -\Delta y \rho_\mathrm{ice}\,g H(x) H'(x) \Delta x$$

$$S_\mathrm{b} = -\rho_\mathrm{ice}\,g H(x)\frac{\mathrm{d}H}{\mathrm{d}x} \rightarrow \frac{S_\mathrm{b}}{\rho_\mathrm{ice}\,g}\int_x^L \mathrm{d}x' = \int_{H(x)}^0 H\,\mathrm{d}H = \frac{1}{2}H(x)^2$$

Implies the profile:

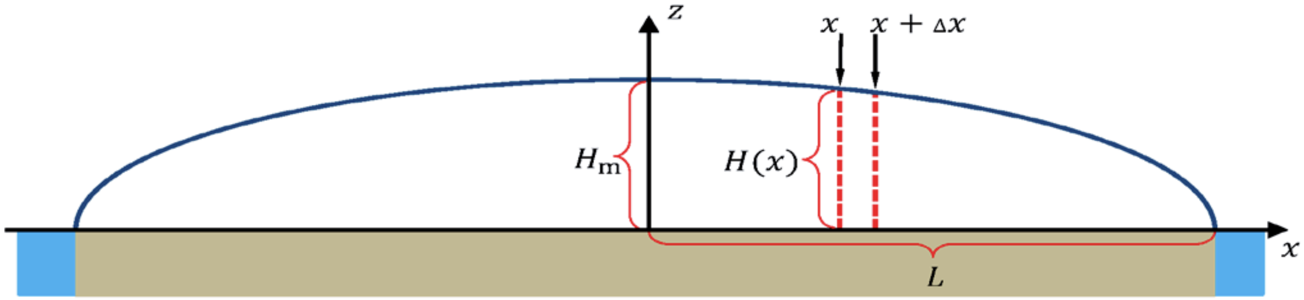
$$H(x) = H_m\sqrt{1-\frac{x}{L}}\,, \qquad H_m = \sqrt{\frac{2S_bL}{\rho_\mathrm{ice}\,g}}$$

Hints given:

$$S_\mathrm{b} \propto H\,\mathrm{d}H/\mathrm{d}x$$
$$H_\mathrm{m} \propto L^{1/2}$$

allows dimensional analysis!

Volume scaling with area



➢ Determine the exponent γ for the volume V_{ice} to area A scaling: $V_{ice} \propto A^{\gamma}$

$$V_{ice} = (5L)2\int_0^L H(x)\mathrm{d}x$$

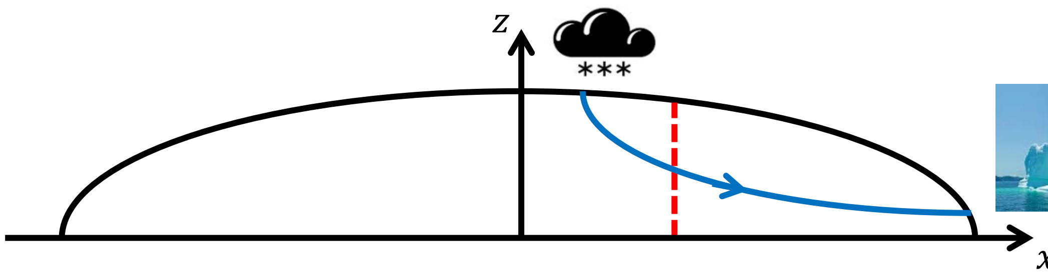
$$= 10LH_m \int_0^L \sqrt{1-\frac{x}{L}}\,\mathrm{d}x$$

$$= 10H_m L^2 \int_0^1 \sqrt{1-\tilde{x}}\,\mathrm{d}\tilde{x}$$

$$= 10H_m L^2 \left[-\tfrac{2}{3}(1-\tilde{x})^{3/2}\right]_0^1$$

$$= \frac{20}{3}H_m L^2 \quad \boxed{\propto L^{5/2}}$$

Dynamical ice sheet

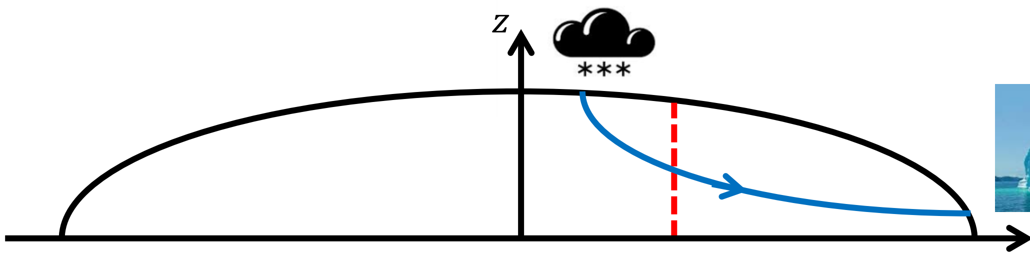


- Ice as a viscous incompressible fluid, which by gravity flows from center to coast.
- Height profile $H(x)$ is maintained in a steady state:

Accumulation of ice due to snow fall in the central region	$\leftrightarrows$	Melting and calving of ice-bergs at the coast

- Consider only the central region $|x| \ll L$ where $H(x) \approx H_\mathrm{m}$, and assume:
  - Ice flows parallel to the x-axis
  - Constant accumulation rate c (m/year)
  - Ice can only leave the glacier by melting near the coast
  - Horizontal ice flow velocity $v_x(x) = \mathrm{d}x/\mathrm{d}t$ is independent of z
  - Vertical ice flow velocity $v_z(z) = \mathrm{d}z/\mathrm{d}t$ is independent of x.
- Mass balance across vertical plane: $\rho c w x = \rho w H_\mathrm{m} v_x(x)$ → $v_x(x) = \dfrac{cx}{H_\mathrm{m}}$

Dynamical ice sheet



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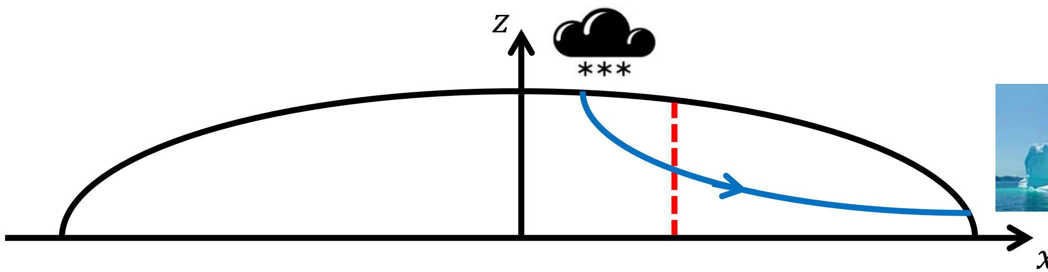
- Mass balance across vertical plane:

$$\frac{\mathrm{d}v_x}{\mathrm{d}x} + \frac{\mathrm{d}v_z}{\mathrm{d}z} = 0 \;\rightarrow\; \frac{\mathrm{d}v_z}{\mathrm{d}z} = -\frac{\mathrm{d}v_x}{\mathrm{d}x} = -\frac{c}{H_\mathrm{m}} \;\rightarrow\; v_z(z) = -\frac{cz}{H_\mathrm{m}}$$

Dynamical ice sheet



- Ice particle initially at $(x_i, H(x_i))$ flows as part of the ice sheet along trajectory $z(x)$

- Found already: $v_x(x) = \frac{cx}{H_\mathrm{m}}$, $v_z(z) = -\frac{cz}{H_\mathrm{m}}$ → *Alternative route...* $z(t) = H_\mathrm{m}\, \mathrm{e}^{-ct/H_\mathrm{m}}$, $x(t) = x_i\, \mathrm{e}^{ct/H_\mathrm{m}}$

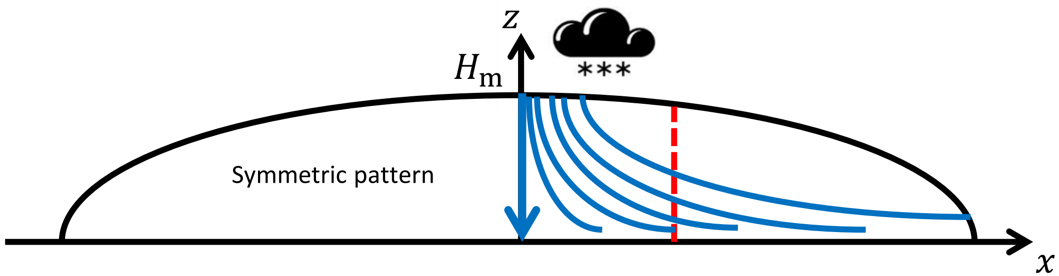
- Construct constant of motion:

$$\frac{\mathrm{d}}{\mathrm{d}t}(xz) = \frac{\mathrm{d}x}{\mathrm{d}t} z + x \frac{\mathrm{d}z}{\mathrm{d}t} = \frac{cx}{H_\mathrm{m}} z - x \frac{cz}{H_\mathrm{m}} = 0 \quad \rightarrow \quad xz = \text{const.}$$

Hyperbolas!

- Use initial condition: $z = H_\mathrm{m} x_i / x$

Chronology of ice layers



- Ice accumulates vertically at the ice divide

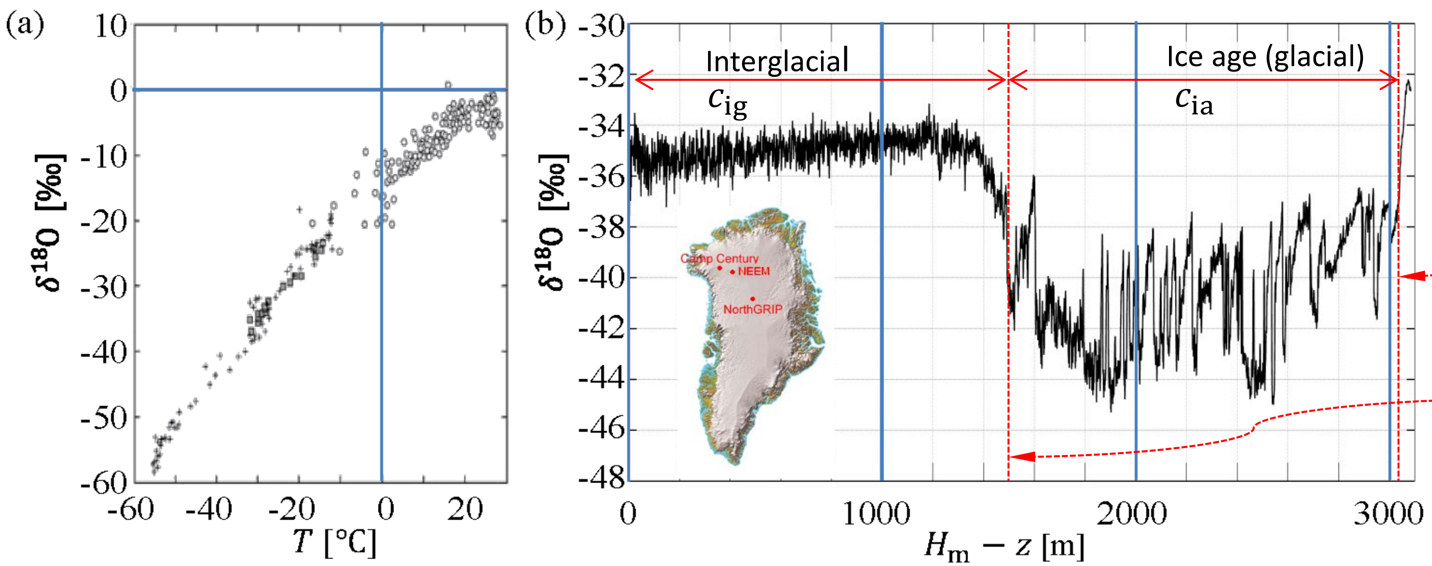
- Estimate the age $\tau(z)$ of the ice at depth $H_\mathrm{m} - z$ from the surface

- Found already: $\dfrac{\mathrm{d}z}{\mathrm{d}t} = v_z(z) = -\dfrac{cz}{H_\mathrm{m}}$ → $z(t) = H_\mathrm{m}\, \mathrm{e}^{-ct/H_\mathrm{m}}$

- Invert this relation and use initial condition: $\tau = \dfrac{H_\mathrm{m}}{c} \ln\left(\dfrac{H_\mathrm{m}}{z}\right)$

Revealing past climate changes

- Data from a drilled ice-core at the ice-divide ($H_m = 3060$ m):



Age: 120,000 year
Depth: 3040 m

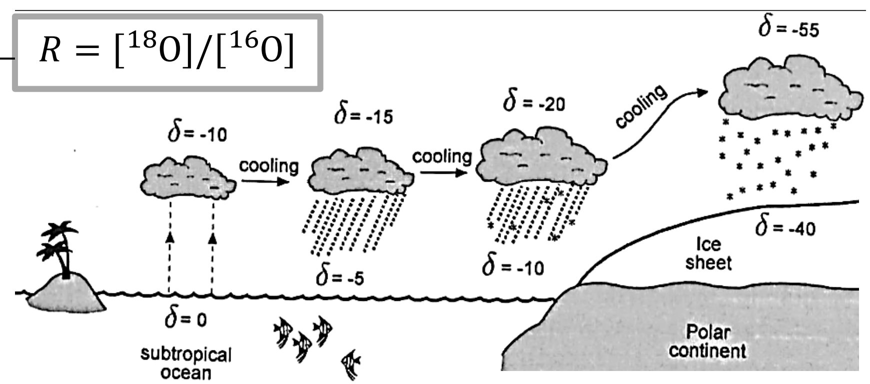
Age: 11,700 year
Depth: 1492 m

- Observation: $\delta^{18}O$ varies approximately linearly with temperature

$$\delta^{18}O = \frac{R_{ice} - R_{ref}}{R_{ref}} 1000 ‰,$$

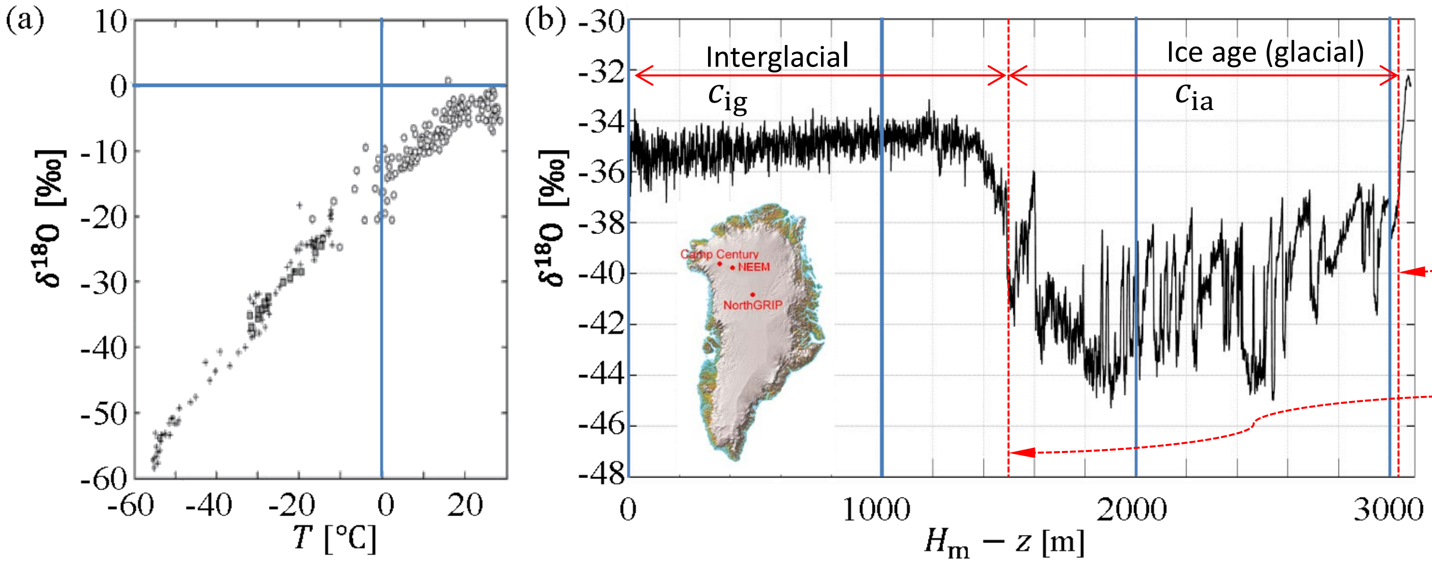
$R = [^{18}O]/[^{16}O]$

Reference ratio in equatorial oceans



Revealing past climate changes

- Data from a drilled ice-core at the ice-divide ($H_\mathrm{m} = 3060$ m):



Age: 120,000 year
Depth: 3040 m

Age: 11,700 year
Depth: 1492 m

- Determine the two accumulation rates c_ig and c_ia

$$c_\mathrm{ig} = \frac{H_\mathrm{m}}{11{,}700\ \mathrm{year}} \ln\left(\frac{H_\mathrm{m}}{H_\mathrm{m} - 1492\ \mathrm{m}}\right) = \boxed{0.16\ \mathrm{m/year}}$$

$$\frac{\mathrm{d}z}{z} = -\frac{c}{H_\mathrm{m}}\mathrm{d}t$$

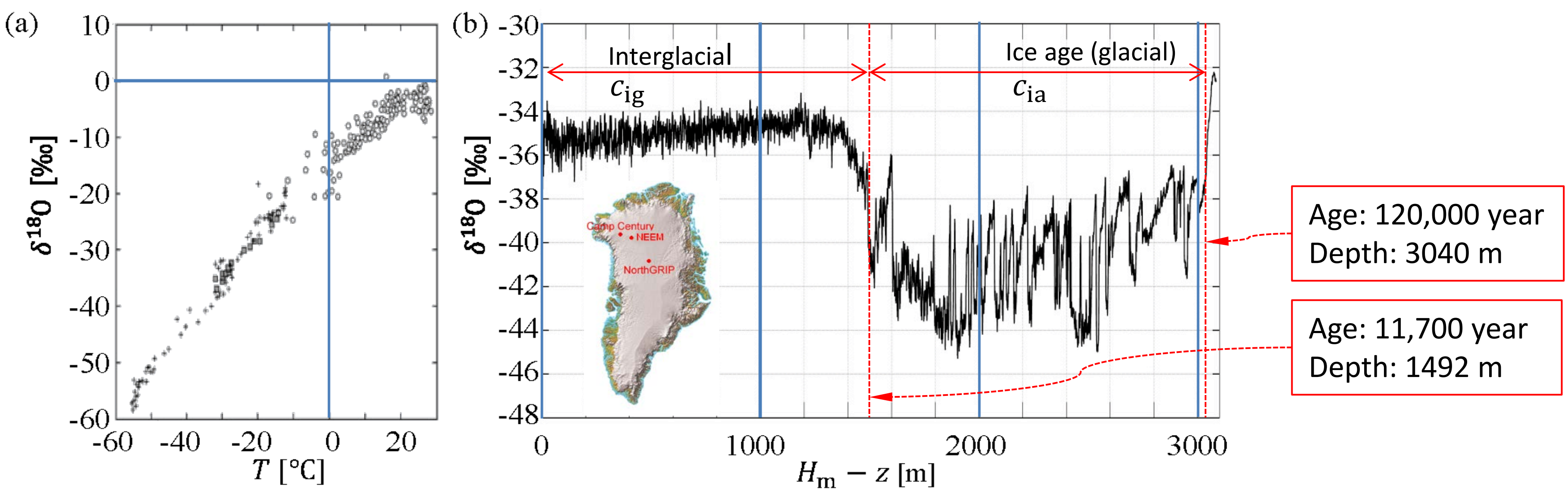
- Integrate up two sections:

$$-H_\mathrm{m} \ln\left(\frac{H_\mathrm{m}}{H_\mathrm{m} - 3040\ \mathrm{m}}\right) = \int_{H_\mathrm{m}}^{3040\ \mathrm{m}} \frac{\mathrm{d}z}{z} = \int_0^{11{,}700\ \mathrm{year}} c_\mathrm{ig}\,\mathrm{d}t + \int_{11{,}700\ \mathrm{year}}^{120{,}000\ \mathrm{year}} c_\mathrm{ia}\,\mathrm{d}t \rightarrow c_\mathrm{ia} = \boxed{0.12\ \mathrm{m/year}}$$

Less precipitation!

Revealing past climate changes

- Data from a drilled ice-core at the ice-divide ($H_m = 3060$ m):



- Temperature change at transition from ice age to interglacial age?

  - Reading off from panel (b): δ^{18}O changes from $-43{,}5$ ‰ to $-34{,}5$ ‰

  - Reading off from panel (a): T changes from -40 °C to -28 °C

$$\Delta T \approx 12 \text{ °C}$$

Sea level rise from melting

- Sea level rise from melting Greenlandic ice-sheet
  - Area of Greenlandic ice sheet $A_\mathrm{G} = 1.71 \times 10^{12}\ \mathrm{m}^2$
  - Global ocean with constant area $A_\mathrm{O} = 3.61 \times 10^{14}\ \mathrm{m}^2$
  - Basal shear stress $S_\mathrm{b} = 100\ \mathrm{kPa}$

$$L = \sqrt{A_\mathrm{G}/10} = 4.14 \times 10^5\ \mathrm{m}$$

- Volume calculated earlier:

$$V_\mathrm{G,ice} = \frac{20}{3} L^{5/2} \sqrt{\frac{2S_\mathrm{b}}{\rho_\mathrm{ice} g}} = 3.46 \times 10^{15}\ \mathrm{m}^3$$

- Volume of the water:

$$V_\mathrm{G,wa} = V_\mathrm{G,ice} \frac{\rho_\mathrm{ice}}{\rho_\mathrm{wa}} = 3.17 \times 10^{15}\ \mathrm{m}^3$$

- Corresponding rise of global ocean:

$$h_\mathrm{G,rise} = \frac{V_\mathrm{G,wa}}{A_\mathrm{O}} = 8.78\ \mathrm{m}$$

Sea level rise from melting

- Sea level rise from melting Greenlandic ice-sheet
  - Area of Greenlandic ice sheet $A_G = 1.71 \times 10^{12}\ \mathrm{m}^2$
  - Global ocean with constant area $A_O = 3.61 \times 10^{14}\ \mathrm{m}^2$
  - Basal shear stress $S_b = 100\ \mathrm{kPa}$

- Sea level rise from melting Antarctic ice-sheet?
  - Antarctica ice sheet is <u>quadratic</u> with area $A_A = 1.21 \times 10^{13}\ \mathrm{m}^2$

- Translate aspect-ratios and volume-area scaling law ($\gamma = 5/4$):

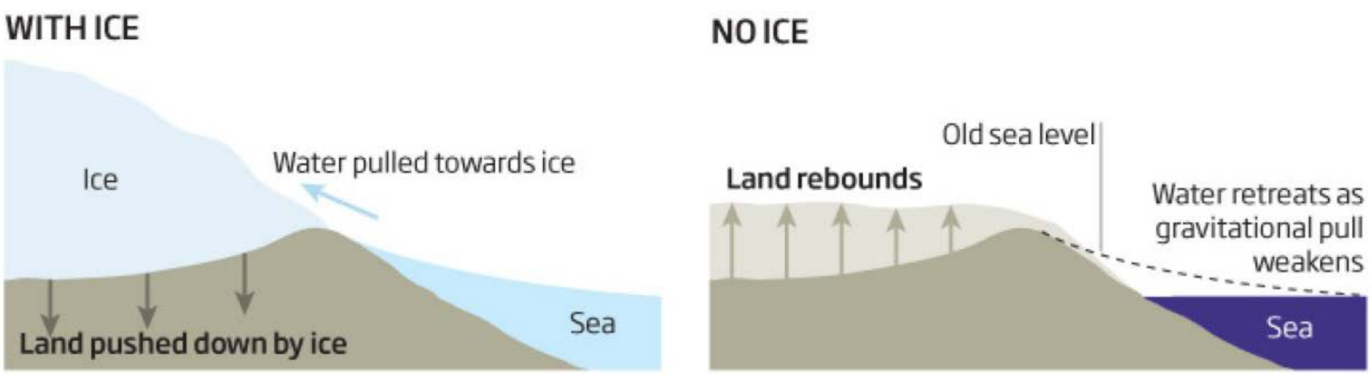
$$\frac{h_{\mathrm{A,rise}}}{h_{\mathrm{G,rise}}} = \frac{V_{\mathrm{A,wa}}}{V_{\mathrm{G,wa}}} = \frac{2}{5}\left(\frac{L_A}{L_G}\right)^{5/2} = \frac{2}{5}\left(\frac{5}{2}\frac{A_A}{A_G}\right)^{5/4} = \left(\frac{5}{2}\right)^{1/4}\left(\frac{A_A}{A_G}\right)^{5/4}$$

- Corresponding rise of global ocean: $h_{\mathrm{A,rise}} = 127\ \mathrm{m}$

High tide from gravitational pull of ice-sheet

- New Scientist 2013:

  "If Greenland's ice were to go completely, then the sea around northern Scotland would subside by more than 3 metres. Around Iceland, it would fall by 10 metres."



- How to estimate this effect?

High tide from gravitational pull of ice-sheet

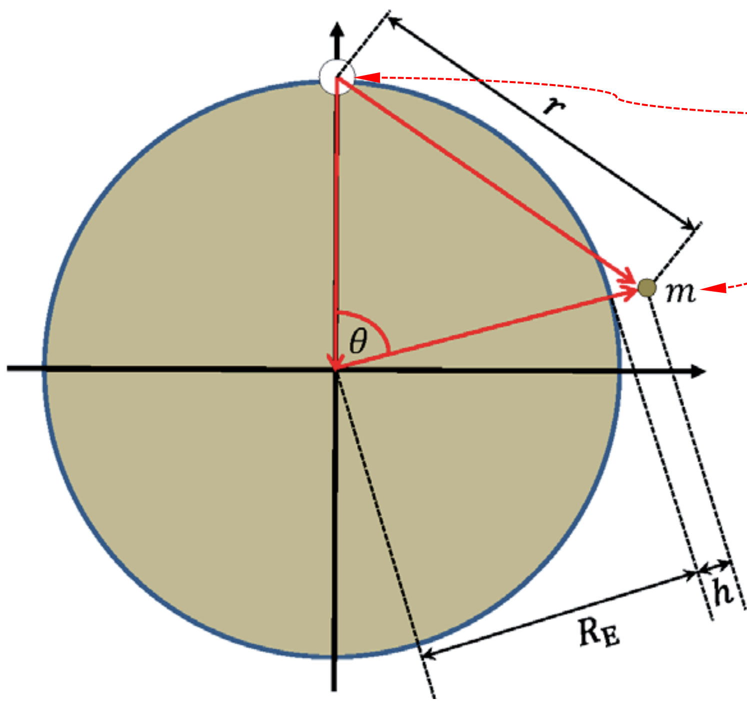
- What is the gravitational effect of the Greenlandic ice-sheet?

  - Causes a high tide on the northern hemisphere

  - Ice mass: $M_{\text{ice}} = V_{\text{G,ice}}\, \rho_{\text{ice}} = 3.17 \times 10^{18}\ \text{kg} = 5.31 \times 10^{-7} m_{\text{E}}$

  - Radius of sphere with ice volume: $R_{\text{ice}} = \left(\frac{3\, V_{\text{G,ice}}}{4\pi}\right)^{1/3} = 93.8\ \text{km}$

- A simple model for estimating the effect:



Ice ball of radius R_{ice} and mass M_{ice}

Test mass m at height h

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High tide from gravitational pull of ice-sheet

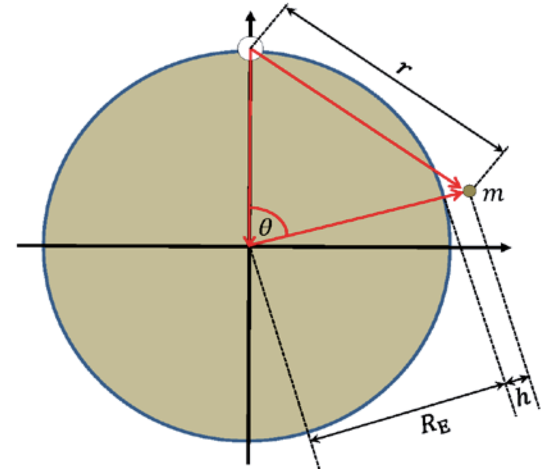
- Total gravitational potential at test mass (*ocean surface = equipotential surface*):

$$U_{\text{tot}} = -\frac{Gm_{\text{E}}m}{R_{\text{E}}+h} - \frac{GM_{\text{ice}}m}{r}$$

$$= -mgR_{\text{E}}\left(\frac{1}{1+h/R_{\text{E}}} + \frac{M_{\text{ice}}/m_{\text{E}}}{r/R_{\text{E}}}\right)$$

$$\approx -mgR_{\text{E}}\left(1 - \frac{h}{R_{\text{E}}} + \frac{M_{\text{ice}}/m_{\text{E}}}{r/R_{\text{E}}}\right)$$

$$g = \frac{Gm_{\text{E}}}{R_{\text{E}}^2}$$



Given: $(1+\mathrm{x})^a \approx 1 + ax,\ \ |ax| \ll 1$

$$h = h_0 + \frac{M_{\text{ice}}/m_{\text{E}}}{r/R_{\text{E}}}R_{\text{E}}$$

$$\approx h_0 + \frac{M_{\text{ice}}/m_{\text{E}}}{2|\sin(\theta/2)|}R_{\text{E}}$$

$$= h_0 + \frac{M_{\text{ice}}/m_{\text{E}}}{2|\sin(S/2R_{\text{E}})|}R_{\text{E}}$$

$$h_0 = R_E + U_{\text{tot}}/(mg)$$

$$r \approx 2R_{\text{E}}|\sin(\theta/2)|$$

$$S = 3500 \text{ km}$$

$$h_{\text{GRL}} - h_{\text{CPH}} \approx 223.5 \text{ m}$$

$$h_{\text{CPH}} - h_{\text{OPP}} \approx 4.5 \text{ m}$$

The IPhO syllabus in relation to the problem

Problem	Syllabus
3.1 Pressure	3. Hydrostatic pressure
3.2 Force balance	1. b) Newton's laws General d) (SI units)
3.3-3.5 Mass balance	3. Continuity law
3.6-3.8 Flow velocities, age vs. depth	*General modelling* *Data analysis*
3.9 Gravitational pull	1. e) The law of gravitation, potential energy and work in a gravitational field